Mathematically speaking…

$\sqrt{-1} = j$

complex number

$4 + 3j$

real part — imaginary part

Euler's Formula

Euler's number

$$e^{j\theta} = \cos\theta + j\sin\theta$$

Sinusoids

Complex exponential function

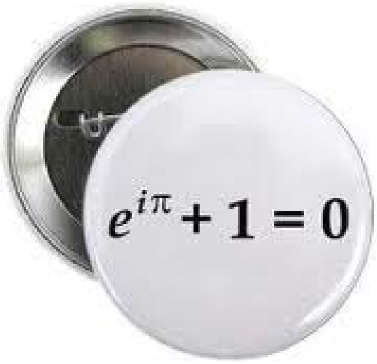
- 15 April 1707 – 18 September 1783
- a Swiss mathematician, physicist, astronomer, logician and engineer
- Made important and influential discoveries in many branches of mathematics

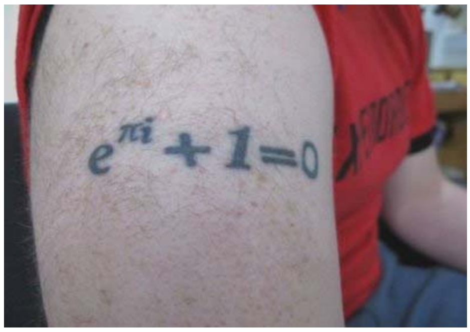
[https://www.youtube.com/watch?v=N321EcSzNas]

The Most Beautiful Equation

Euler's identity (Euler's equation)

Relate the three fundamental constants e, π and i.

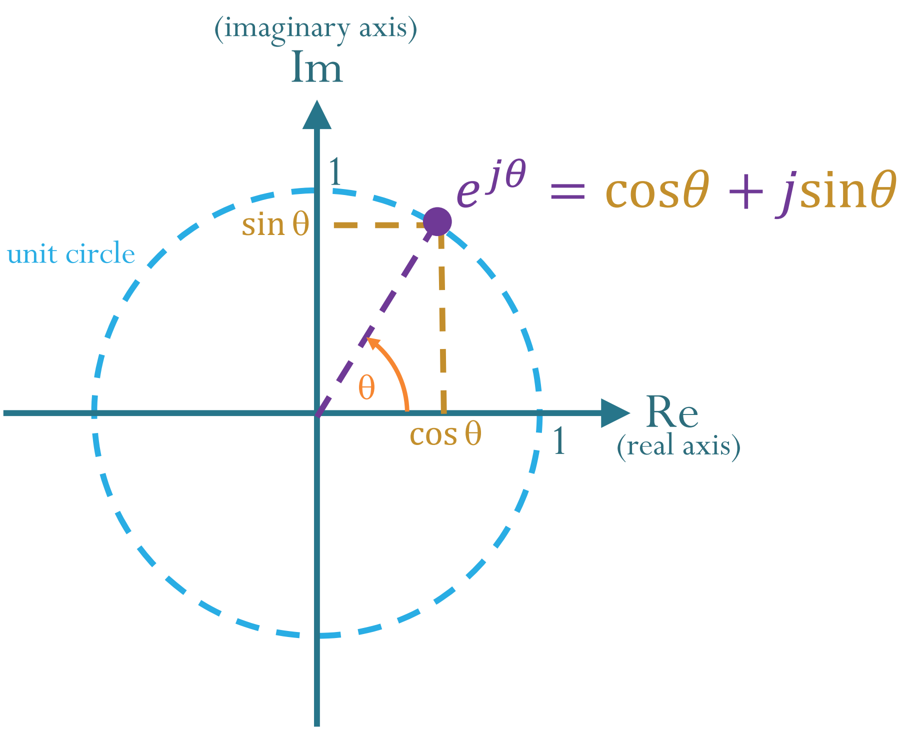




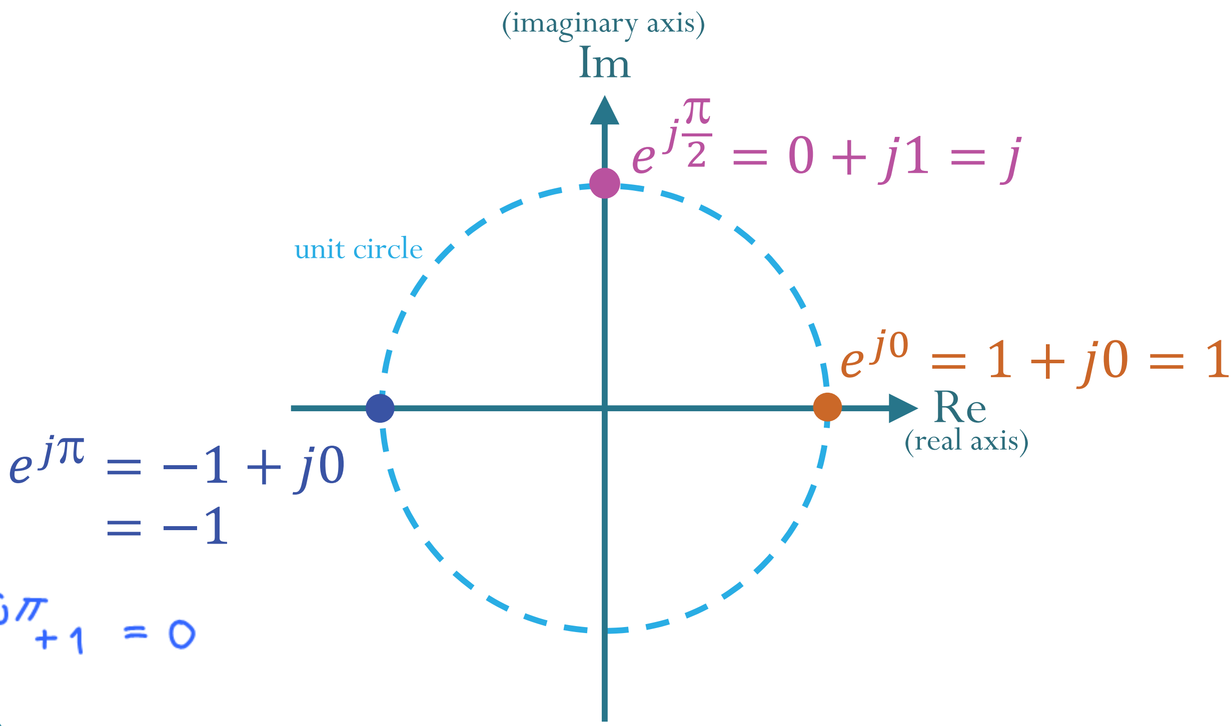
Fact: When mathematicians describe equations as beautiful, they are not lying. Brain scans show that their minds respond to beautiful equations in the same way other people respond to great paintings or masterful music.

[http://www.scientificamerican.com/article/equations-are-art-inside-a-mathematicians-brain/]

Euler's Formula on the Complex Plane

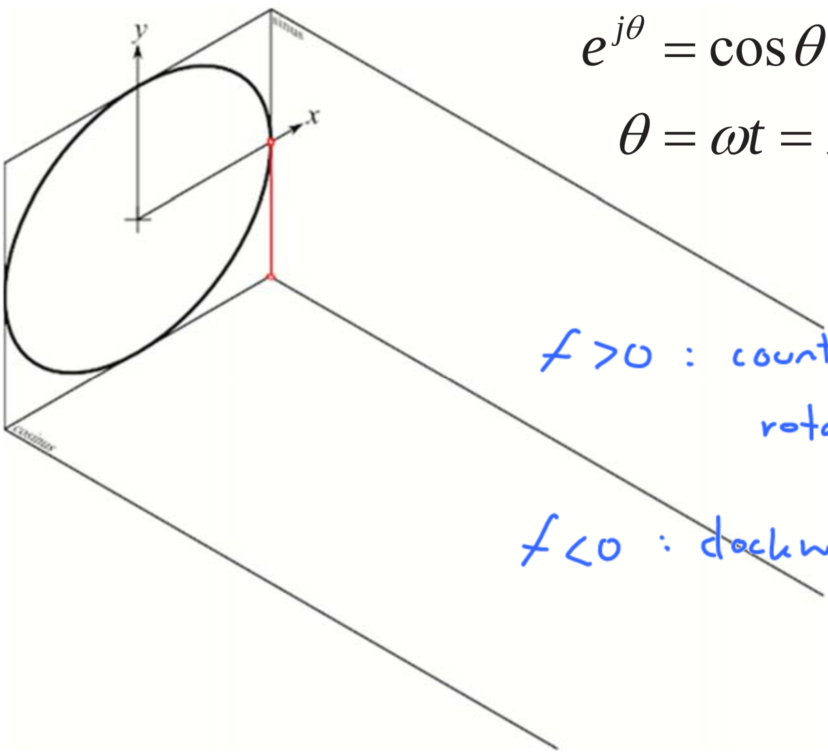


Euler's Formula on the Complex Plane



$e^{j\pi} + 1 = 0$

Rotating Vector in a Complex Plane



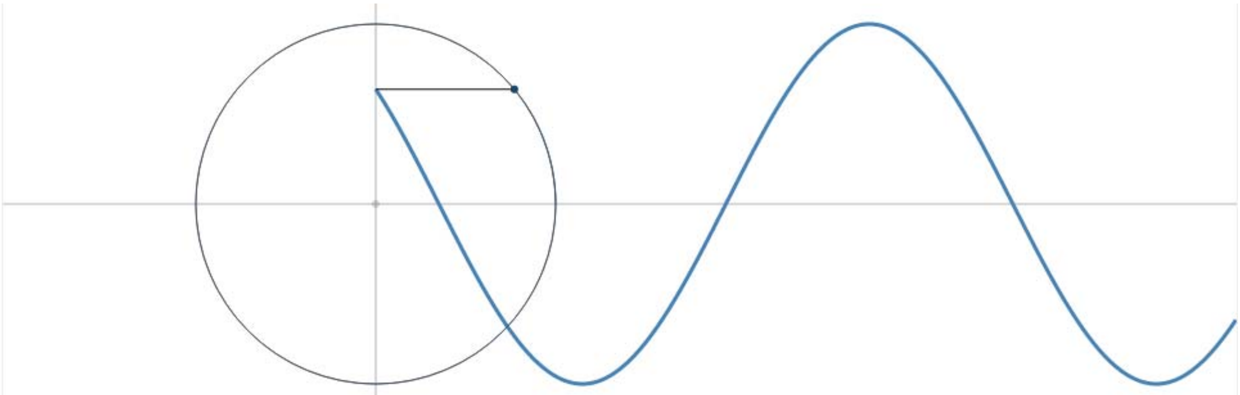
$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$\theta = \omega t = 2\pi ft$$

$f > 0$: counter-clockwise rotation

$f < 0$: clockwise rotation

$$\mathrm{Im}\{e^{j\theta}\} = \sin\theta$$



[http://bl.ocks.org/jinroh/7524988]

Euler's Formula

$$e^{j\theta} = \cos\theta + j\sin\theta$$

Complex exponential

$$\cos(A) = \mathrm{Re}\{e^{jA}\} = \frac{1}{2}\left(e^{jA} + e^{-jA}\right)$$

$$\sin(A) = \mathrm{Im}\{e^{jA}\} = \frac{1}{2j}\left(e^{jA} - e^{-jA}\right)$$

$$e^{jA} = \cos A + j\sin A$$

$$e^{j(-A)} = \cos(-A) + j\sin(-A) = \cos(A) - j\sin(A)$$

Euler's Formula

$\cos(-\alpha) = \frac{1}{2}\left(e^{j(-\alpha)} + e^{-j(-\alpha)}\right)$
$= \frac{1}{2}\left(e^{-j\alpha} + e^{j\alpha}\right)$
$= \cos(\alpha)$

$$e^{j\theta} = \cos\theta + j\sin\theta$$

Complex exponential

$$\cos(A) = \mathrm{Re}\{e^{jA}\} = \frac{1}{2}\left(e^{jA} + e^{-jA}\right)$$

$$\sin(A) = \mathrm{Im}\{e^{jA}\} = \frac{1}{2j}\left(e^{jA} - e^{-jA}\right)$$

$$\cos(-x) = \cos(x)$$

$$\cos\left(x - \frac{\pi}{2}\right) = \sin(x)$$

$$2\cos^2 x = 1 + \cos(2x)$$

$$2\sin^2 x = 1 - \cos(2x)$$

$$2\sin(x)\cos(x) = \sin(2x)$$

$$\frac{d}{dx}\sin x = \cos x$$

$$\cos(x)\cos(y) = \frac{1}{2}\left(\cos(x+y) + \cos(x-y)\right)$$

(product-to-sum formula)

(Continuous-Time) Fourier Transform

$$g(t) = \int_{-\infty}^{\infty} G(f)e^{j2\pi ft}df \overset{\mathcal{F}}{\rightleftharpoons} G(f) = \int_{-\infty}^{\infty} g(t)e^{-j2\pi ft}dt$$

sinusoids

Complex exponential: $e^{j2\pi ft} = \cos(2\pi ft) + j\sin(2\pi ft)$

The relationship on the left is simply a **decomposition** of the signal $g(t)$ into a **linear combination** of (potentially infinitely many) $e^{j2\pi ft}$ components at different frequencies.

(Continuous-Time) Fourier Transform

$$g(t) = \int_{-\infty}^{\infty} G(f) e^{j2\pi ft} df \overset{\mathcal{F}}{\rightleftharpoons} G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi ft} dt$$

From the decomposition point of view, the value of $G(f)$ at a particular frequency f is simply the **weight** (scaling/coefficient) which tells how much $e^{j2\pi ft}$ component there is in $g(t)$.

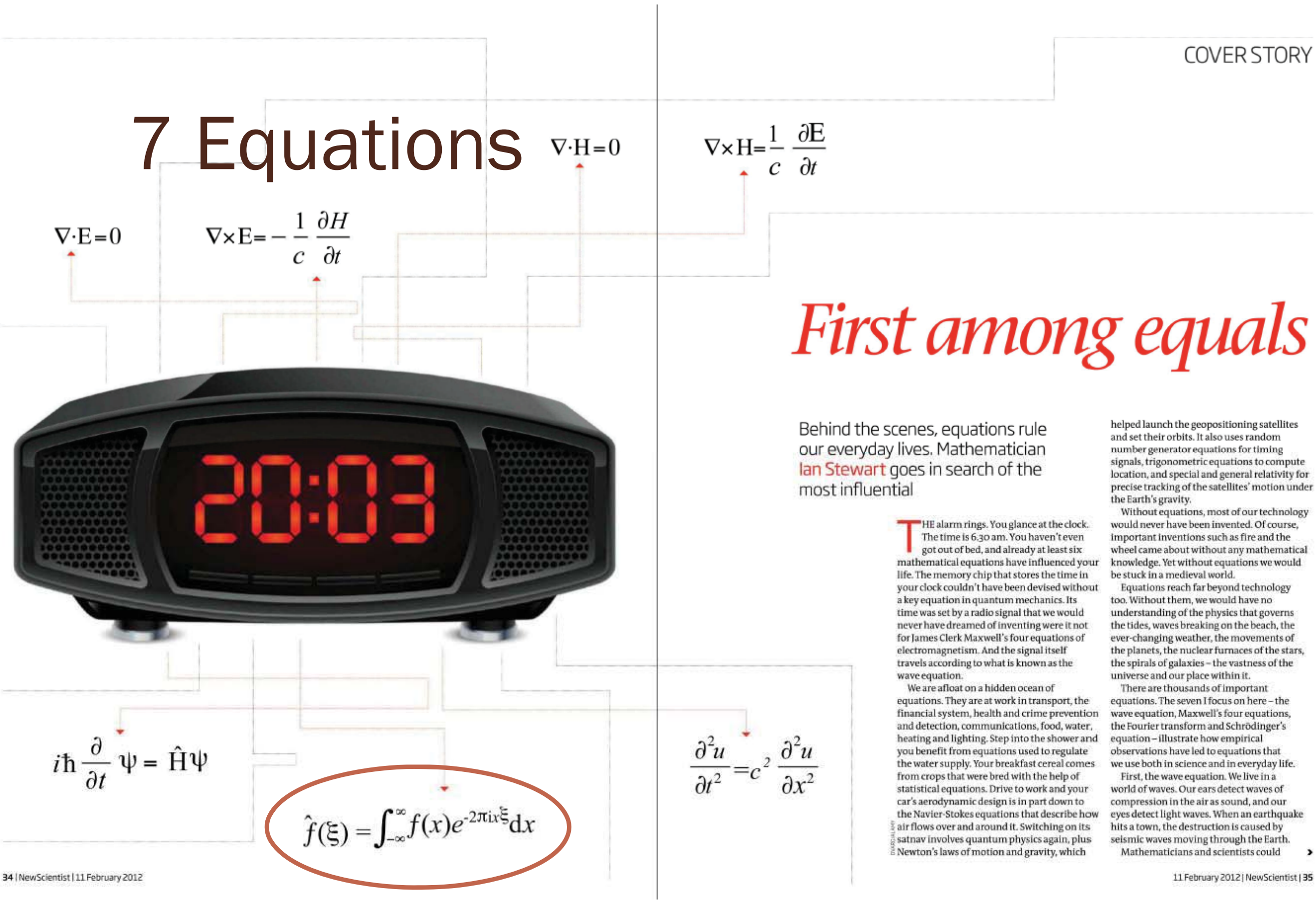
By the **orthogonality** among complex exponential functions, the value of $G(f)$ at a particular frequency f can be calculated by the formula above.

This coefficient $G(f)$ considered as a function of frequency is the **Fourier transform** of our signal.

7 Equations

that changed the world

… and still rule everyday life



7 Equations

$\nabla \cdot E = 0$ $\quad$ $\nabla \times E = -\frac{1}{c}\frac{\partial H}{\partial t}$ $\quad$ $\nabla \cdot H = 0$ $\quad$ $\nabla \times H = \frac{1}{c}\frac{\partial E}{\partial t}$

$i\hbar \frac{\partial}{\partial t}\psi = \hat{H}\psi$ $\quad$ $\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx$ $\quad$ $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

First among equals

Behind the scenes, equations rule our everyday lives. Mathematician Ian Stewart goes in search of the most influential

THE alarm rings. You glance at the clock. The time is 6.30 am. You haven't even got out of bed, and already at least six mathematical equations have influenced your life. The memory chip that stores the time in your clock couldn't have been devised without a key equation in quantum mechanics. Its time was set by a radio signal that we would never have dreamed of inventing were it not for James Clerk Maxwell's four equations of electromagnetism. And the signal itself travels according to what is known as the wave equation.

We are afloat on a hidden ocean of equations. They are at work in transport, the financial system, health and crime prevention and detection, communications, food, water, heating and lighting. Step into the shower and you benefit from equations used to regulate the water supply. Your breakfast cereal comes from crops that were bred with the help of statistical equations. Drive to work and your car's aerodynamic design is in part down to the Navier-Stokes equations that describe how air flows over and around it. Switching on its satnav involves quantum physics again, plus Newton's laws of motion and gravity, which helped launch the geopositioning satellites and set their orbits. It also uses random number generator equations for timing signals, trigonometric equations to compute location, and special and general relativity for precise tracking of the satellites' motion under the Earth's gravity.

Without equations, most of our technology would never have been invented. Of course, important inventions such as fire and the wheel came about without any mathematical knowledge. Yet without equations we would be stuck in a medieval world.

Equations reach far beyond technology too. Without them, we would have no understanding of the physics that governs the tides, waves breaking on the beach, the ever-changing weather, the movements of the planets, the nuclear furnaces of the stars, the spirals of galaxies – the vastness of the universe and our place within it.

There are thousands of important equations. The seven I focus on here – the wave equation, Maxwell's four equations, the Fourier transform and Schrödinger's equation – illustrate how empirical observations have led to equations that we use both in science and in everyday life.

First, the wave equation. We live in a world of waves. Our ears detect waves of compression in the air as sound, and our eyes detect light waves. When an earthquake hits a town, the destruction is caused by seismic waves moving through the Earth.

Mathematicians and scientists could ›

(Continuous-Time) Fourier Transform

Time Domain $\qquad\qquad$ **Frequency Domain**

$$g(t) = \int_{-\infty}^{\infty} G(f) e^{j2\pi f t} df \;\underset{\text{inverse transform}}{\overset{\text{direct transform}}{\underset{\longleftarrow}{\overset{\mathcal{F}}{\longrightarrow}}}}\; G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi f t} dt$$

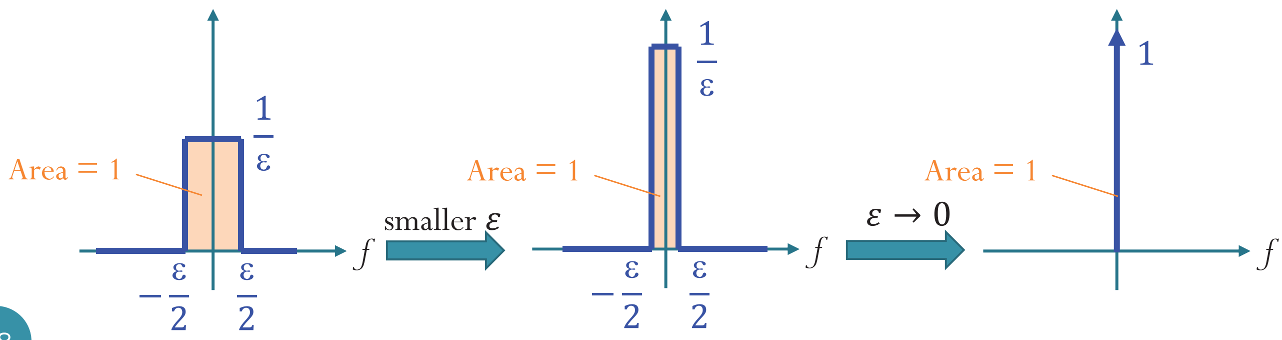
Capital letter is used to represent corresponding signal in the frequency domain.

Signals in this form is "easy" to work with under LTI system.

$$g(0) = \int_{-\infty}^{\infty} G(f) df \qquad\qquad G(0) = \int_{-\infty}^{\infty} g(t) dt$$

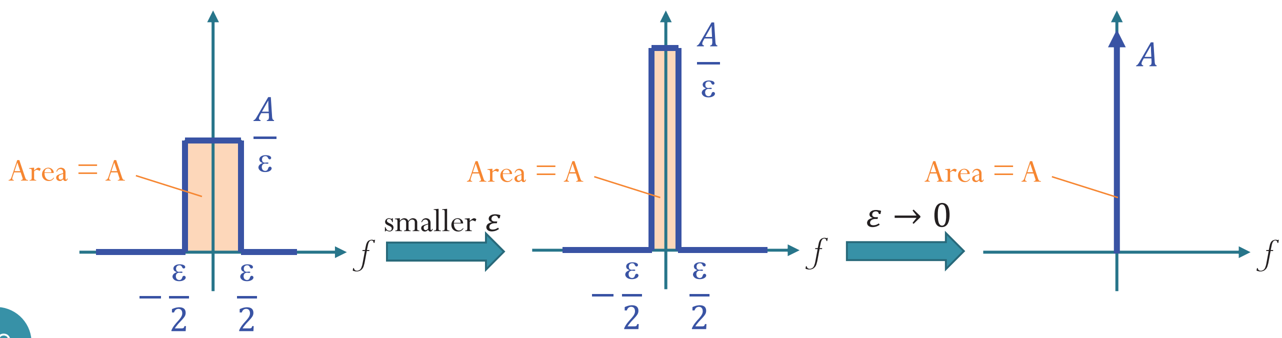
Delta function δ(f)

- (Dirac) delta function or (unit) impulse function
- Usually depicted as a vertical arrow at the origin
- Not a true function
  - Undefined at $f = 0$
- Intuitively we may visualize $\delta(f)$ as an infinitely tall, infinitely narrow rectangular pulse of **unit area**



Aδ(f)

- (Dirac) delta function or (unit) impulse function
- Usually depicted as a vertical arrow at the origin
- Not a true function
  - Undefined at $f = 0$
- Intuitively we may visualize $A\delta(f)$ as an infinitely tall, infinitely narrow rectangular pulse of **area A**



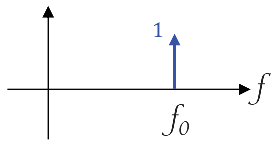
Fourier Transform Pairs (1)

Time Domain | Frequency Domain

$$g(t)=\int_{-\infty}^{\infty}G(f)e^{j2\pi ft}df \overset{\mathcal{F}}{\rightleftharpoons} G(f)=\int_{-\infty}^{\infty}g(t)e^{-j2\pi ft}dt$$

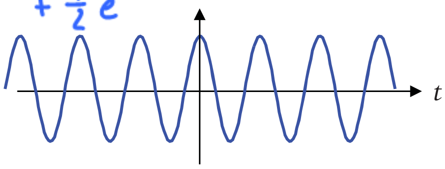
$\cos(A) = \frac{1}{2}\left(e^{jA}+e^{-jA}\right)$

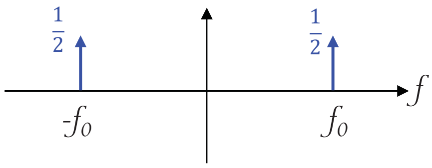
$$e^{j2\pi f_0 t} \overset{\mathcal{F}}{\rightleftharpoons} \delta(f-f_0)$$



$$\frac{e^{j2\pi f_0 t}+e^{-j2\pi f_0 t}}{2} = \cos(2\pi f_0 t) \overset{\mathcal{F}}{\rightleftharpoons} \frac{1}{2}\delta(f-(-f_0))+\frac{1}{2}\delta(f-f_0)$$

$= \frac{1}{2}e^{j2\pi f_0 t} + \frac{1}{2}e^{j2\pi(-f_0)t}$

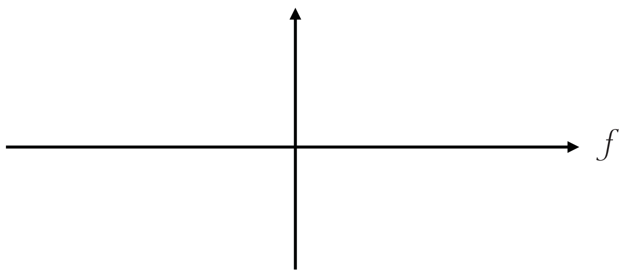




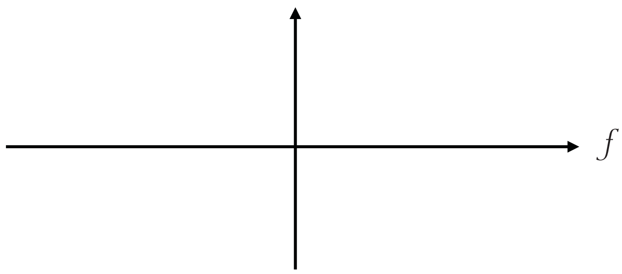
$\sin(A) = \frac{1}{2j}\left(e^{jA}-e^{-jA}\right) = \frac{1}{2j}e^{jA} + \left(-\frac{1}{2j}e^{-jA}\right)$

Practice Problems (A Revisit)

$\cos(200\pi t) \xrightarrow{\mathcal{F}}$



$\cos(200t) \xrightarrow{\mathcal{F}}$

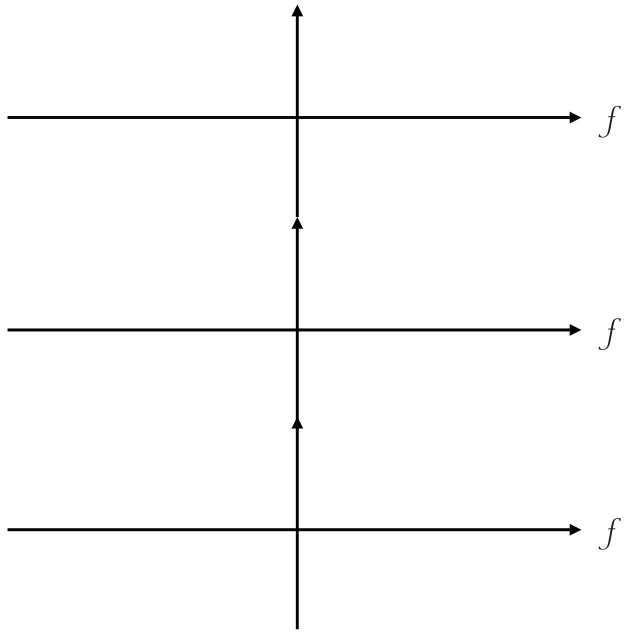


Practice Problems (More)

$$\cos(200\pi t)+\cos(400\pi t) \xrightarrow{\mathcal{F}}$$

$$\cos^2(200\pi t) \xrightarrow{\mathcal{F}}$$

$$\cos(200\pi t)\times\cos(400\pi t) \xrightarrow{\mathcal{F}}$$



Fourier Transform of Symbolic Expression in MATLAB

```
function G = fourierf(g)
syms f
G = simplify(subs(fourier(g),'w',2*pi*f));
end
```

```
>> syms t; g = exp(1j*2*pi*5*t);
>> G = fourierf(g)
G =
dirac(f - 5)
```

```
>> syms t; g = cos(2*pi*5*t);
>> G = fourierf(g)
G =
dirac(f - 5)/2 + dirac(f + 5)/2
>> pretty(G)

  dirac(f - 5)    dirac(f + 5)
  ------------ + ------------
       2               2
```

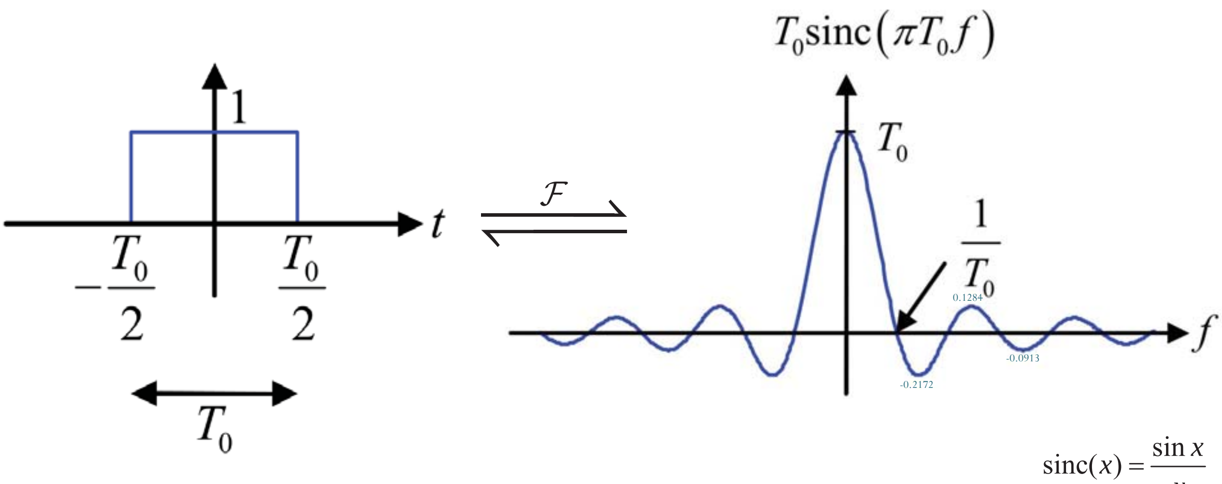
```
>> syms t f0; g = cos(2*pi*f0*t);
>> G = fourierf(g)
G =
dirac(f + f0)/2 + dirac(f - f0)/2
>> pretty(G)

  dirac(f + f0)    dirac(f - f0)
  ------------- + -------------
        2                2
```

Fourier Transform Pairs (2)

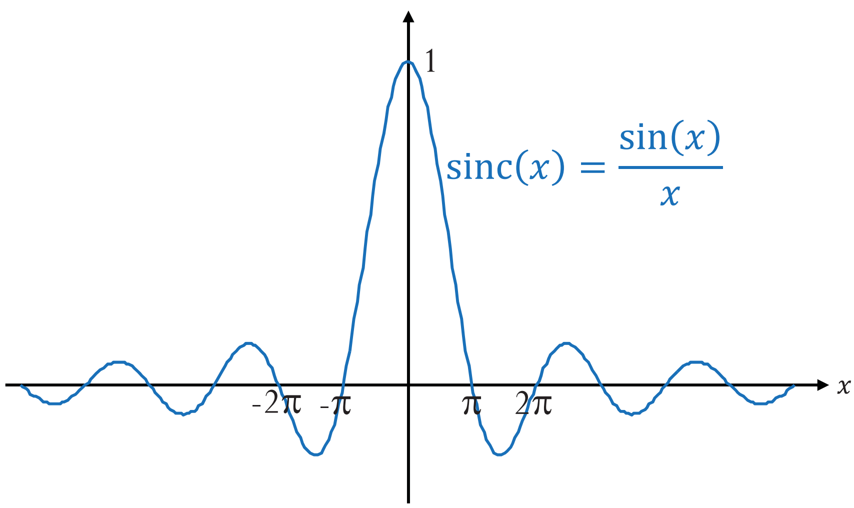
Time Domain — Frequency Domain

$$g(t)=\int_{-\infty}^{\infty}G(f)e^{j2\pi ft}df \overset{\mathcal{F}}{\rightleftharpoons} G(f)=\int_{-\infty}^{\infty}g(t)e^{-j2\pi ft}dt$$



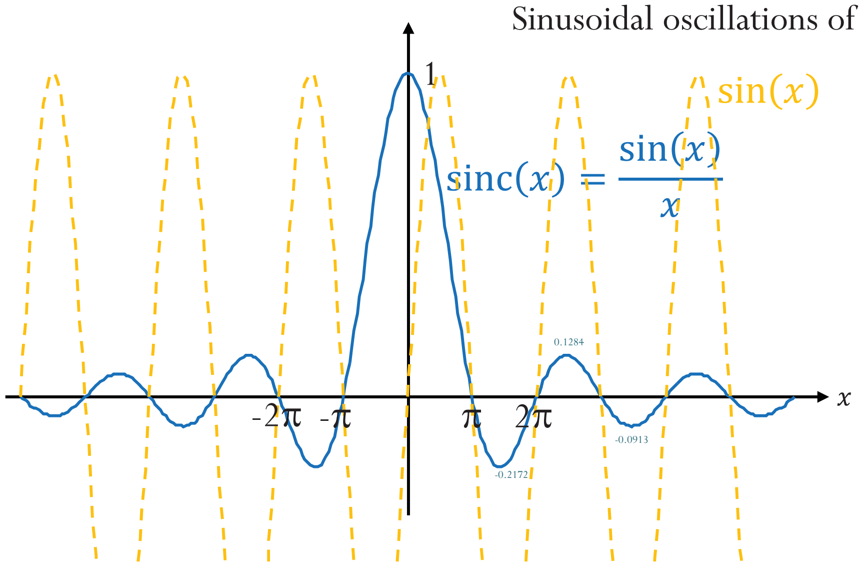
$$\text{sinc}(x)=\frac{\sin x}{x}$$

sinc function



sinc function

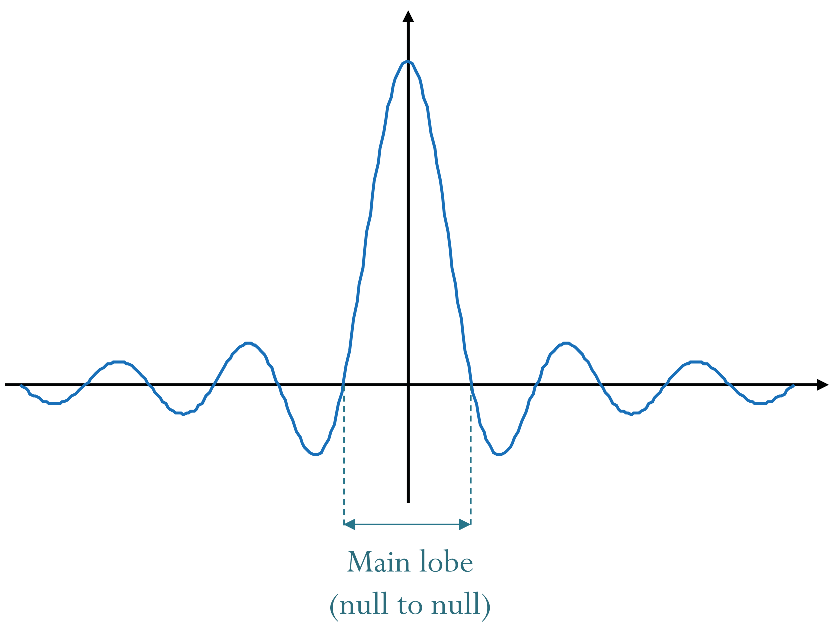
Sinusoidal oscillations of period 2π



Zero crossings are at all non-zero integer multiples of π because $\sin(x) = 0$.

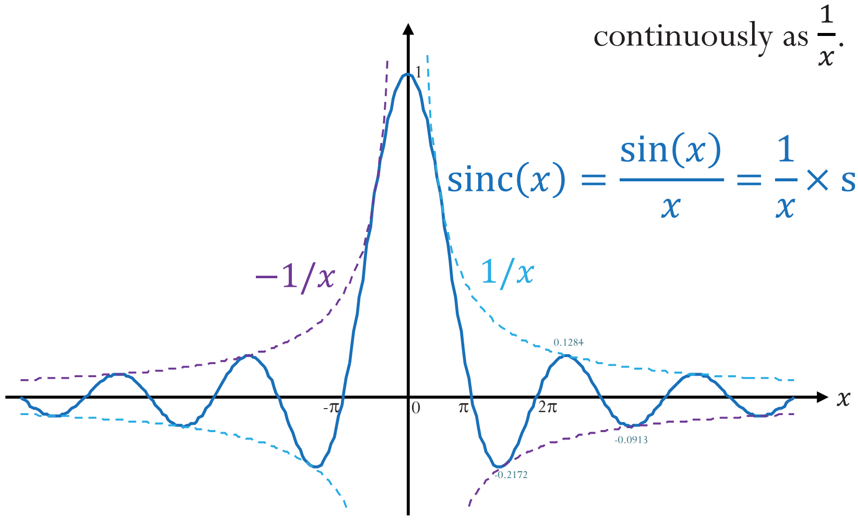
As $x \to 0$, we have $\frac{0}{0}$. Using L'Hospital's Rule, we set $\text{sinc}(0) \equiv 1$.

sinc function



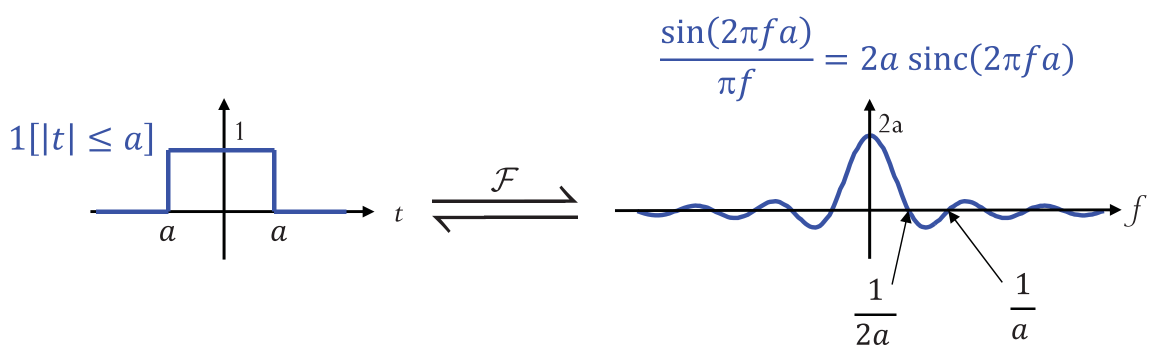
sinc function

Amplitude of $\sin(x)$ decreases continuously as $\frac{1}{x}$.

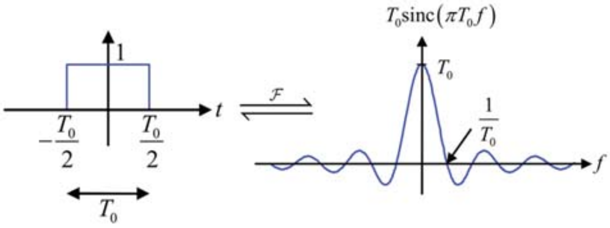


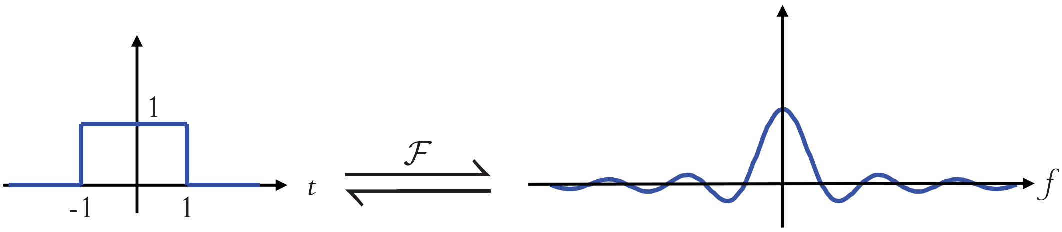
Fourier Transform of Symbolic Rectangular Function in MATLAB

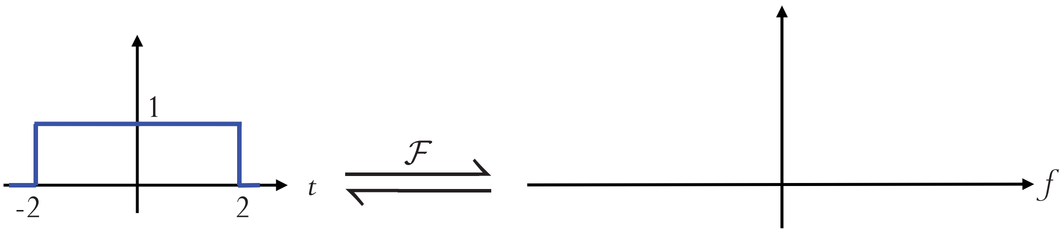
```
>> syms a t
>> g = rectangularPulse(-a,a,t)
g =
rectangularPulse(-a, a, t)
>> G = fourierf(g)
G =
sin(2*pi*a*f)/(pi*f)
```



Practice Problems







THE PROCEEDINGS OF
THE INSTITUTION OF ELECTRICAL ENGINEERS

EDITED UNDER THE SUPERINTENDENCE OF W. K. BRASHER, C.B.E., M.A., M.I.E.E., SECRETARY

VOL. 99. PART III (RADIO AND COMMUNICATION ENGINEERING), No. 58. MARCH 1952

621.391 : 519.21 : 530.162

Paper No. 1239
RADIO SECTION

INFORMATION THEORY AND INVERSE PROBABILITY IN TELECOMMUNICATION

By P. M. WOODWARD, B.A., and I. L. DAVIES, M.A., Graduate.

(*The paper was first received* 11*th October, and in revised form* 10*th December*, 1951.)

Sam
theore
greater

$$f(t) \equiv \sum_r f(r/2W)\,\text{sinc}\,(2Wt - r) \quad . \quad . \quad . \quad (24)$$

where sinc x is an abbreviation for the function (sin πx)/πx. This function occurs so often in Fourier analysis and its applications that it does seem to merit some notation of its own. Its most important properties are that it is zero when x is a whole number but unity when x is zero, and that

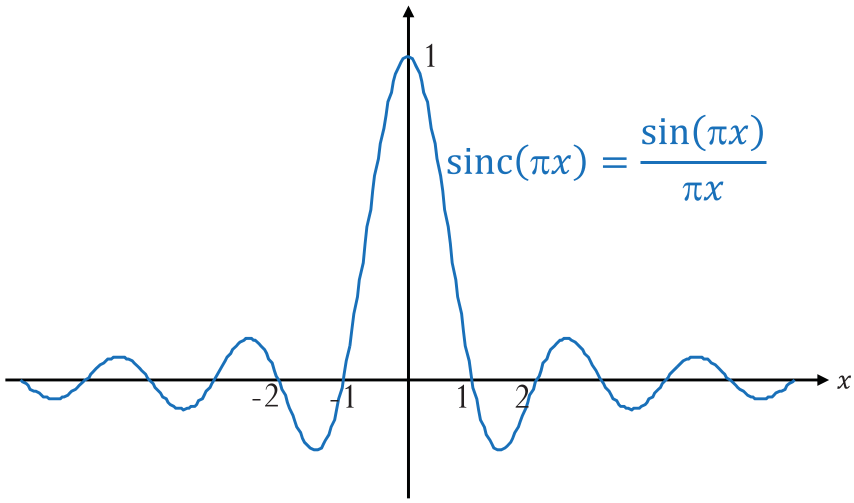
Normalized sinc function

$$\int_{-\infty}^{\infty} \text{sinc}\,x\, dx = 1$$

and
$$\int_{-\infty}^{\infty} \text{sinc}\,(x-r)\,\text{sinc}\,(x-s)\,dx = \begin{cases} 1, & r = s \\ 0, & r \neq s \end{cases}$$

r and s both being integers. The importance of the identity (24)

Normalized sinc function

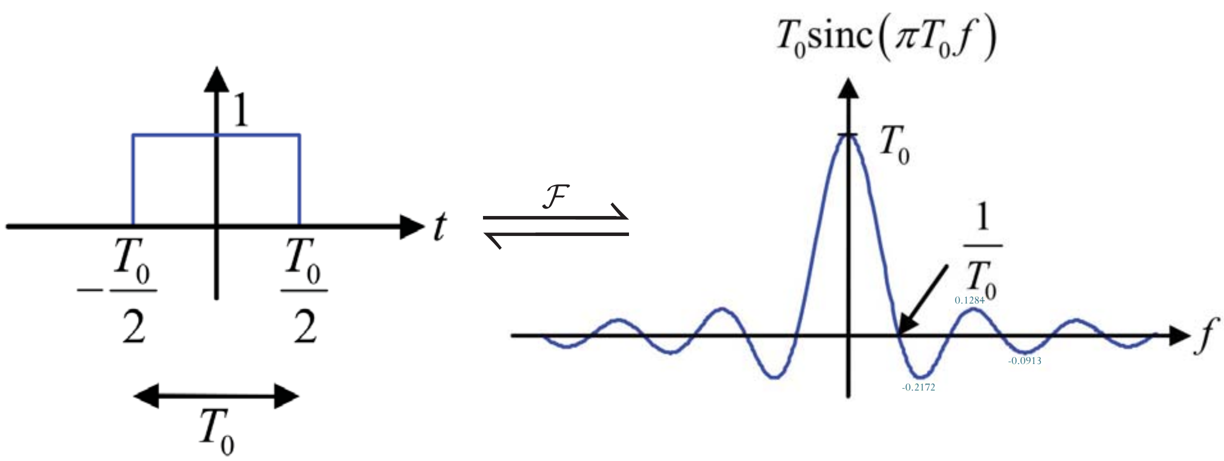


Its zero crossings are at non-zero integer values of its argument.

Fourier Transform Pairs (2)

Time Domain | Frequency Domain

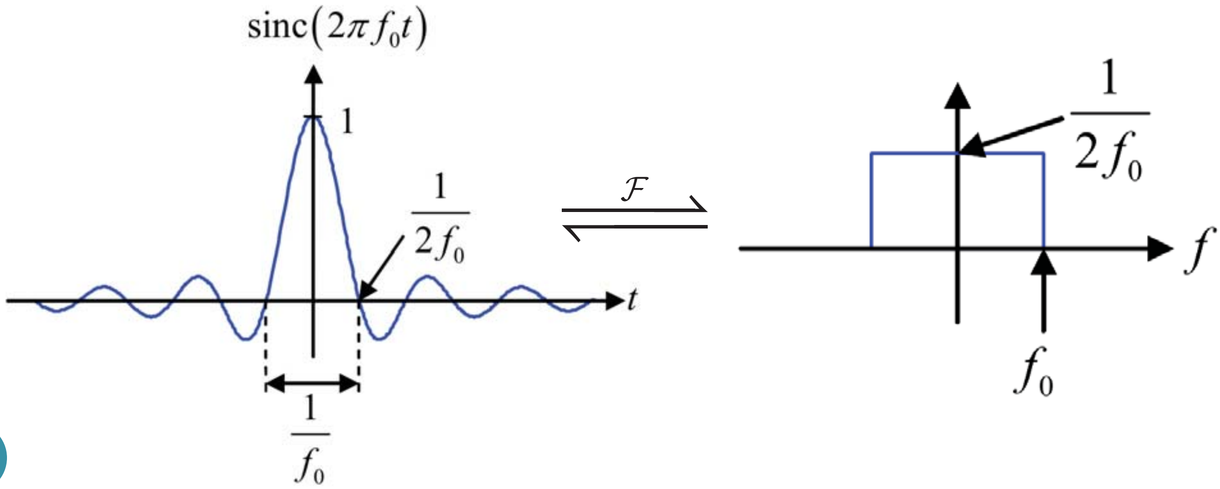
$$g(t)=\int_{-\infty}^{\infty} G(f)e^{j2\pi ft}df \overset{\mathcal{F}}{\rightleftharpoons} G(f)=\int_{-\infty}^{\infty} g(t)e^{-j2\pi ft}dt$$



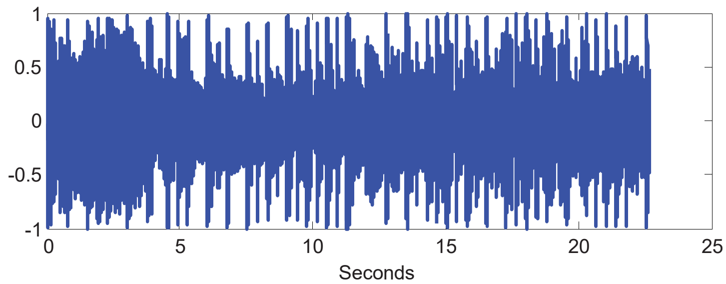
Fourier Transform Pairs (3)

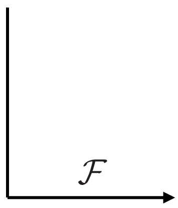
Time Domain | Frequency Domain

$$g(t) = \int_{-\infty}^{\infty} G(f) e^{j2\pi ft} df \overset{\mathcal{F}}{\rightleftharpoons} G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi ft} dt$$

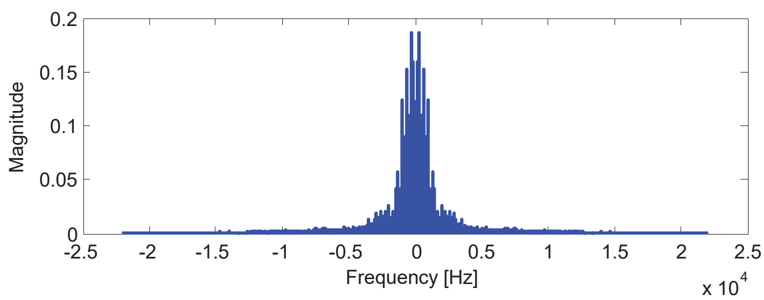


More realistic signal...





plotspect.m



plotspect.m

```
% plotspec(x,t) plots the spectrum of the signal x
% whose values are sampled at time (in seconds) specified in t
function plotspect(x,t)
N=length(x);                                % length of the signal x
Ts = t(2)-t(1);                             % find the sampling interval
ssf=((-N/2):(N/2-1))/(Ts*N);                % frequency vector
fx=Ts*fft(x(1:N));                          % do DFT/FFT
fxs=fftshift(fx);                           % shift it for plotting
subplot(2,1,1);
set(plot(t,x),'LineWidth',1.5);             % plot the waveform
xlabel('Seconds');                          % label the axes
subplot(2,1,2);
set(plot(ssf,abs(fxs)),'LineWidth',1.5);    % plot magnitude spectrum
xlabel('Frequency [Hz]'); ylabel('Magnitude')    % label the axes
```

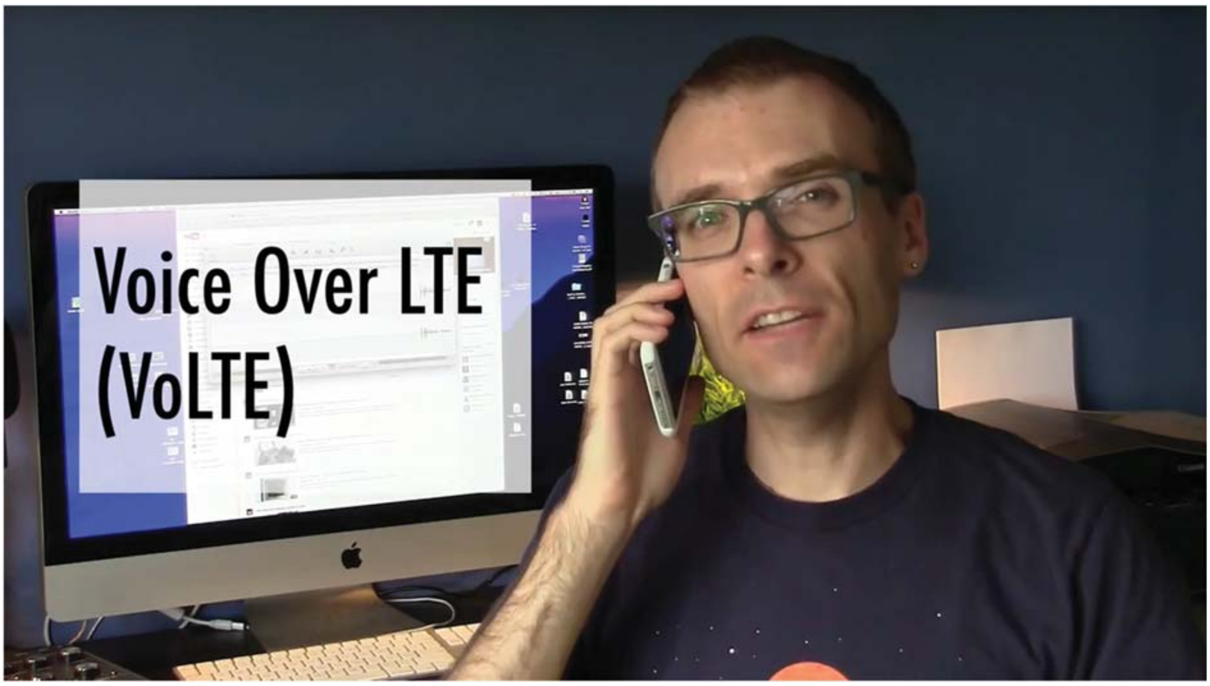

Phone/Cellphone (Muffled) Audio

[https://www.youtube.com/watch?v=CPFiufYmtAc]

PILOT'S ALPHABET

- International Radiotelephony Spelling Alphabet.
- Pilots use it so that essential letter combinations can be easily understood by individuals transmitting and receiving voice messages.

VoLTE Audio

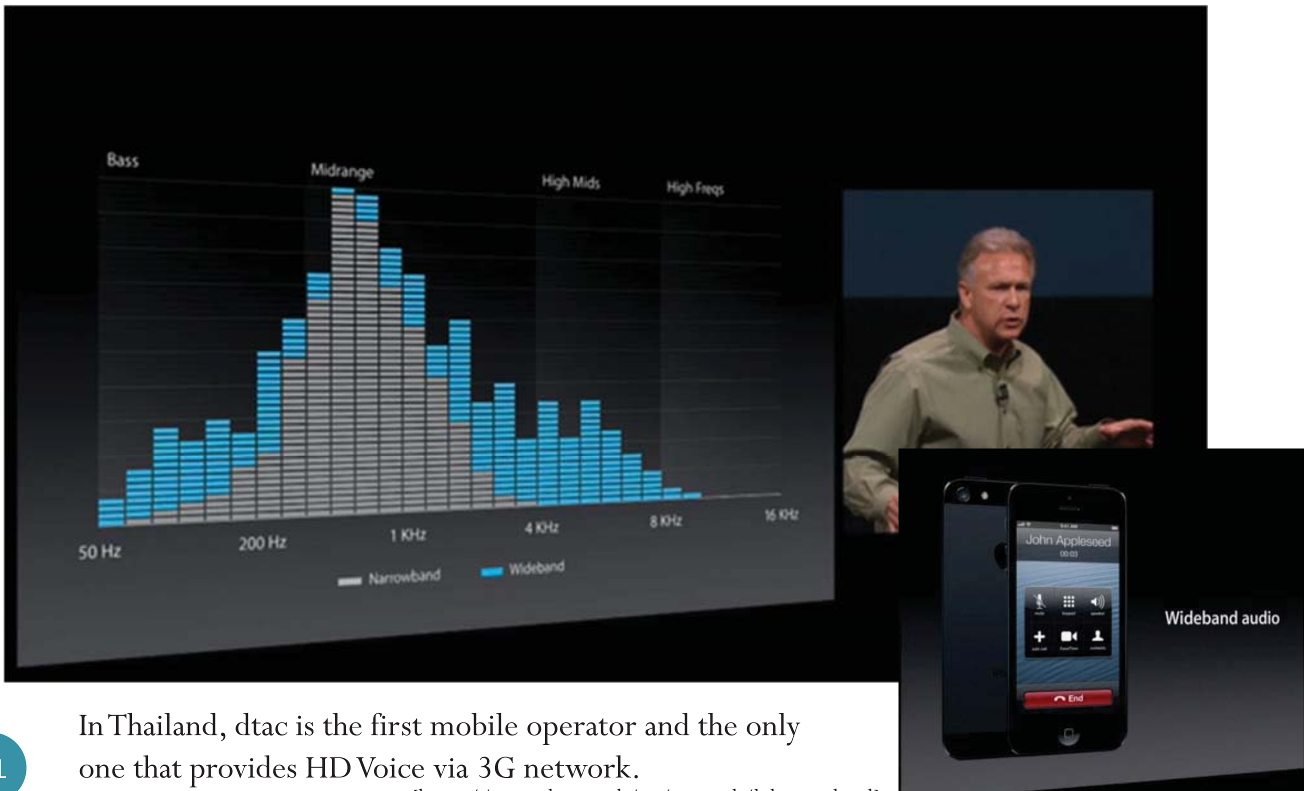


[https://www.youtube.com/watch?v=CPFiufYmtAc]

VoLTE Audio: AMR-WB

[https://www.youtube.com/watch?v=CPFiufYmtAc]

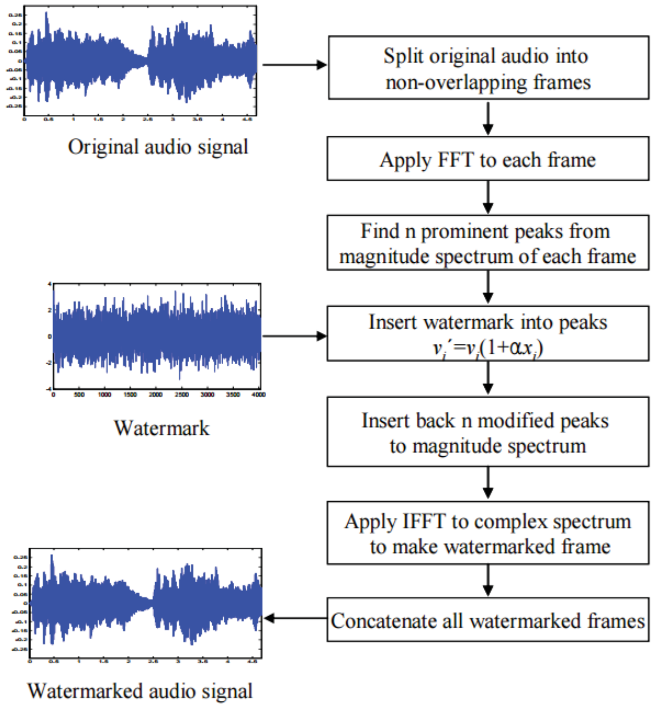
iPhone 5 supports HD Voice in the form of AMR-WB over 3G (UMTS)



In Thailand, dtac is the first mobile operator and the only one that provides HD Voice via 3G network.

[https://www.dtac.co.th/en/network/hd-voice.html]

Digital Audio Watermarking



[http://www.sersc.org/journals/IJSIA/vol5_no2_2011/3.pdf]